

The automorphism group of the sandwich inverse symmetric semigroup

G.M. Kudryavtseva, G.Y. Tsyaputa

Abstract

The structure of the automorphism group of the sandwich semigroup \mathcal{IS}_n is described in terms of standard group constructions.

1 Introduction

Let X and Y be two nonempty sets, S be the set of all maps from X to Y . Fix some $\alpha : Y \rightarrow X$ and define the multiplication in S in the following way: $\varphi \circ \psi = \varphi\alpha\psi$ (the composition of the maps is from the left to the right). The action defined by this rule is associative. Ljapin in [?, p.353] set the problem of investigation of the properties of this semigroup depending on the restrictions on S and α .

Magill in [2] has considered this problem in the case when X and Y are topological spaces and the maps are continuous. In particular, under the assumption that α is onto he has described the automorphisms of such a semigroup and has determined the isomorphism criterion of two such semigroups.

Sullivan in [3] has proved that if $|Y| \leq |X|$ then Ljapin's semigroup embeds into transformation semigroup on the set $X \cup \{a\}$, $a \notin X$.

The important case is if $X = Y$. In this case one considers T_X , the transformation semigroup on X , $\alpha \in T_X$. Symons in [4] has established the isomorphism criterion for such semigroups and has described their automorphism groups.

The problem of the description of automorphism group is one of the most important in the process of study of a certain algebraic structure. In particular, much attention has been devoted to the description of automorphism groups of different semigroups (see, for example, [5], [6] and the references therein).

In the present paper we study the structure of the automorphism group of the sandwich finite inverse symmetric semigroup \mathcal{IS}_n of all partial injective transformations of set $N = \{1, \dots, n\}$ with the sandwich element $a \in \mathcal{IS}_n$ (denote it by $(\mathcal{IS}_n, *a)$). The main result is analogous to the one on the automorphism group of the sandwich semigroup of all transformations obtained by Symons in [4].

2 The main theorem

For an element $a \in \mathcal{IS}_n$ denote by $\text{dom}(a)$ the domain of a , and by $\text{im}(a)$ the image of a . The cardinality of the set $\text{im}(a)$ is called the *rank* of a and is denoted by $\text{rank}(a)$. If $\text{dom}(a) = \{x_1, \dots, x_k\}$ and $a(x_i) = y_i$ for all $1 \leq i \leq k$ then we write $a = \begin{pmatrix} x_1 & \dots & x_k \\ y_1 & \dots & y_k \end{pmatrix}$.

For a subset $A \subseteq N$ denote by \overline{A} the complement ($N \setminus A$) of A . If $a \in \mathcal{IS}_n$ and $M \subseteq N$ then $a|_M$ denotes the restriction of a to M .

Denote by $E(S)$ the set of idempotents of the semigroup S . Recall (see [7]) that the relation $f \leq h \Leftrightarrow fh = hf = f$ defines the *natural partial order* on $E(S)$.

Finally for arbitrary K denote by $\mathcal{S}(K)$ the full symmetric group on K . Other notions, which are used in the paper without definitions, can be found, for example, in [7].

We will use the following fact first proved in [8].

Theorem 1. *Semigroups $(\mathcal{IS}_n, *_{a_1})$ and $(\mathcal{IS}_n, *_{a_2})$ are isomorphic if and only if $\text{rank}(a_1) = \text{rank}(a_2)$.*

Hence, up to isomorphism one can study only sandwich semigroups $(\mathcal{IS}_n, *)$ with sandwich elements e such that $e \in E(\mathcal{IS}_n)$. From now on we fix idempotent e such that e acts identically on set $A = \{1, 2, \dots, k\}$, $A \subseteq N$, and denote $(\mathcal{IS}_n, *_{e_1})$ just by $(\mathcal{IS}_n, *)$.

Proposition 1. *The lattice of idempotents $E(\mathcal{IS}_n, *)$ is isomorphic to the lattice $\mathcal{B}(A)$ of the subsets of A .*

Proof. In [8] it is proved that element $a \in \mathcal{IS}_n$ is idempotent in $(\mathcal{IS}_n, *)$ if and only if a is the idempotent in \mathcal{IS}_n and $\text{dom}(a) \subseteq A$. Hence the map $f \mapsto \text{dom}(f)$ defines the isomorphism between the partially ordered set $E(\mathcal{IS}_n, *)$ and the lattice $\mathcal{B}(A)$. \square

Corollary 1. The sandwich idempotent e is the maximal element of the lattice of idempotents in $(\mathcal{IS}_n, *)$. In particular, for arbitrary automorphism $\varphi \in \text{Aut}(\mathcal{IS}_n, *)$ we have $\varphi(e) = e$.

Let S be a semigroup, $a \in S$. Element a is called *decomposable* in S provided that $a = bc$ for some $b, c \in S$, otherwise a is called *indecomposable*.

Let P be the set of all decomposable elements in $(\mathcal{IS}_n, *)$, and Q be the set of all indecomposable elements. For every $a \in Q$ consider the sets

$$\begin{aligned} M_1(a) &= \{x \in \text{dom}(a) \mid x \in A, a(x) \in A\}, \\ M_2(a) &= \{x \in \text{dom}(a) \mid x \in A, a(x) \in \overline{A}\}, \\ M_3(a) &= \{x \in \text{dom}(a) \mid x \in \overline{A}, a(x) \in A\}, \\ M_4(a) &= \{x \in \text{dom}(a) \mid x \in \overline{A}, a(x) \in \overline{A}\}. \end{aligned}$$

Obviously $\text{dom}(a) = M_1(a) \cup M_2(a) \cup M_3(a) \cup M_4(a)$. For $a, b \in Q$ set $a \sim b$ provided that $M_i(a) = M_i(b)$, $1 \leq i \leq 3$, and $a(x) = b(x)$ for all $x \in M_1(a) \cup M_2(a) \cup M_3(a)$. If either a or b does not belong to Q then set $a \sim b$ provided that $a = b$.

Lemma 1. *The element $a \in \mathcal{IS}_n$ is decomposable in $(\mathcal{IS}_n, *)$ if and only if $\text{rank}(a) \leq k$.*

Proof. Let a be decomposable in $(\mathcal{IS}_n, *)$. Then $a = b * c$ for some $b, c \in \mathcal{IS}_n$, therefore $\text{rank}(a) = \text{rank}(bec) \leq \text{rank}(e) = k$.

Conversely, let $\text{rank}(a) = l \leq k$ and $a = \begin{pmatrix} x_1 & \cdots & x_l \\ y_1 & \cdots & y_l \end{pmatrix}$. The direct calculation shows that $a = b * c$, where $b = \begin{pmatrix} x_1 & \cdots & x_l \\ 1 & \cdots & l \end{pmatrix}$, $c = \begin{pmatrix} 1 & \cdots & l \\ y_1 & \cdots & y_l \end{pmatrix}$. \square

Lemma 2. *The relation \sim is a congruence on $(\mathcal{IS}_n, *)$.*

Proof. Let $a \sim a_1$ and $b \sim b_1$. Show that $a * b \sim a_1 * b_1$. Since $\text{rank}(a * b) \leq k$ and $\text{rank}(a_1 * b_1) \leq k$, using Lemma 1 we get $a * b, a_1 * b_1 \in P$. Hence by the definition of \sim we have to prove that $a * b = a_1 * b_1$. In fact, $a(x) \in A$, $a_1(x) = a(x)$, and $b(a(x)) = b_1(a(x))$ for arbitrary $x \in \text{dom}(a * b)$. Hence $(a * b)(x) = b(a(x)) = b_1(a(x)) = b_1(a_1(x)) = (a_1 * b_1)(x)$.

Analogously, for every $x \in \text{dom}(a_1 * b_1)$ we obtain $(a_1 * b_1)(x) = (a * b)(x)$. \square

Theorem 2. *Let $(\mathcal{IS}_n, *)$ be the sandwich semigroup \mathcal{IS}_n with sandwich idempotent e and let $\text{dom}(e) = A$. Let further P_1, \dots, P_m be the list of equivalence classes of the relation \sim on \mathcal{IS}_n . Then*

$$\text{Aut}(\mathcal{IS}_n, *) = \left(\bigoplus_{i=1}^m \mathcal{S}(P_i) \right) \times \left(\mathcal{S}(A) \times \mathcal{S}(\bar{A}) \times \mathcal{S}(\bar{A}) \right).$$

We split the proof into several lemmas.

Lemma 3. $\left(\bigoplus_{i=1}^m \mathcal{S}(P_i) \right) < \text{Aut}(\mathcal{IS}_n, *)$.

Proof. The statement follows from the definition of \sim and Lemma 2. \square

Denote by \mathcal{G} the group $(\mathcal{S}(A) \times \mathcal{S}(\bar{A}) \times \mathcal{S}(\bar{A}))$. For arbitrary $g \in \mathcal{S}(A)$ and $h \in \mathcal{S}(\bar{A})$ we denote by ${}_g h$ the element (g, h) in $(\mathcal{S}(A) \times \mathcal{S}(\bar{A}))$. It is obvious that ${}_g h$ can be regarded as the element in \mathcal{IS}_n via the identification with its image under the natural embedding: $(\mathcal{S}(A) \times \mathcal{S}(\bar{A})) \hookrightarrow \mathcal{IS}_n$.

Lemma 4. *For arbitrary $(g, h_1, h_2) \in \mathcal{G}$ the map*

$$\tau_{(g, h_1, h_2)} : a \mapsto {}_g h_1^{-1} a {}_g h_2$$

*is an automorphism of $(\mathcal{IS}_n, *)$, moreover the map $(g, h_1, h_2) \xrightarrow{\varphi} \tau_{(g, h_1, h_2)}$ is the monomorphism from \mathcal{G} to $\text{Aut}(\mathcal{IS}_n, *)$.*

Proof. Set $\tau = \tau_{(g, h_1, h_2)}$. It is enough to show that $\tau(a * b) = \tau(a) * \tau(b)$ for arbitrary $a, b \in \mathcal{IS}_n$. By the construction of ${}_g h_1$ and ${}_g h_2$ we get ${}_g h_1^{-1} * {}_g h_2 = {}_g h_1^{-1} e {}_g h_2 = e$, hence

$$\tau(a * b) = {}_g h_1^{-1} a * b {}_g h_2 = {}_g h_1^{-1} a e b {}_g h_2 = {}_g h_1^{-1} a {}_g h_2 * {}_g h_1^{-1} b {}_g h_2 = \tau(a) * \tau(b).$$

\square

We identify \mathcal{G} with $\varphi(\mathcal{G})$ and consider \mathcal{G} as a subgroup of $\text{Aut}(\mathcal{IS}_n, *)$.

Lemma 5.

$$\left(\mathcal{S}(A) \times \mathcal{S}(\overline{A}) \times \mathcal{S}(\overline{A})\right) \cap \left(\bigoplus_{i=1}^m \mathcal{S}(P_i)\right) = \varepsilon,$$

where ε is the identical automorphism of $(\mathcal{IS}_n, *)$.

Proof. Let $\pi \in \left(\mathcal{S}(A) \times \mathcal{S}(\overline{A}) \times \mathcal{S}(\overline{A})\right) \cap \left(\bigoplus_{i=1}^m \mathcal{S}(P_i)\right)$. Take arbitrary $t, s \in N$ and $a = \begin{pmatrix} t \\ s \end{pmatrix} \in \mathcal{IS}_n$. Then $a \in P$. This and $\pi \in \bigoplus_{i=1}^m \mathcal{S}(P_i)$ imply $\pi(a) = a$. Since $\pi \in \mathcal{S}(A) \times \mathcal{S}(\overline{A}) \times \mathcal{S}(\overline{A})$ it follows that $\pi = \tau_{(g, h_1, h_2)}$ for some $g \in \mathcal{S}(A)$, $h_1, h_2 \in \mathcal{S}(\overline{A})$. Then $\pi(a) = \pi \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} {}_g h_1(t) \\ {}_g h_2(s) \end{pmatrix}$. Hence, ${}_g h_1(t) = t$ and ${}_g h_2(s) = s$, therefore ${}_g h_1 = {}_g h_2 = \varepsilon$.
Hence $\pi(a) = {}_g h_1^{-1} a {}_g h_2 = a$ for arbitrary $a \in \mathcal{IS}_n$, and $\pi = \varepsilon$. \square

Lemma 6. Let $\pi \in \text{Aut}(\mathcal{IS}_n, *)$ and $a \in P$. Then $\text{rank}(\pi(a)) = \text{rank}(a)$.

Proof. Proposition 1 implies that $\text{rank}(\pi(f)) = \text{rank}(f)$ for any idempotent $f \in E(\mathcal{IS}_n, *)$. Let $a \in P$. For arbitrary idempotent $f \in E(\mathcal{IS}_n, *)$ we have

$$\mathcal{IS}_n * f * \mathcal{IS}_n = \{a \in \mathcal{IS}_n : \text{rank}(a) \leq \text{rank}(f)\}.$$

Hence a is contained in the main two-sided ideal generated by f if and only if $\text{rank}(f) \geq \text{rank}(a)$. This yields

$$\text{rank}(a) = \min\{k : a \in \mathcal{IS}_n * f * \mathcal{IS}_n \text{ and } f \text{ is idempotent of rank } k\}.$$

It follows that

$$\text{rank}(\pi(a)) = \min\{k : \pi(a) \in \mathcal{IS}_n * q * \mathcal{IS}_n \text{ and } q \text{ is idempotent of rank } k\} = \text{rank}(a).$$

\square

Consider the sets:

$$\text{Ann}_L(\mathcal{IS}_n, *) = \{x \in \mathcal{IS}_n : x * a = 0 \text{ for all } a \in \mathcal{IS}_n\},$$

$$\text{Ann}_R(\mathcal{IS}_n, *) = \{x \in \mathcal{IS}_n : a * x = 0 \text{ for all } a \in \mathcal{IS}_n\},$$

$$\text{and } \text{Ann}(\mathcal{IS}_n, *) = \text{Ann}_L(\mathcal{IS}_n, *) \cap \text{Ann}_R(\mathcal{IS}_n, *).$$

The following lemma is obvious.

Lemma 7. 1) $x \in \text{Ann}_L(\mathcal{IS}_n, *)$ if and only if $\text{im}(x) \in \overline{A}$,

2) $x \in \text{Ann}_R(\mathcal{IS}_n, *)$ if and only if $\text{dom}(x) \in \overline{A}$,

3) $x \in \text{Ann}(\mathcal{IS}_n, *)$ if and only if $\text{im}(x) \in \bar{A}$ and $\text{dom}(x) \in \bar{A}$.

Lemma 8. $\text{Aut}(\mathcal{IS}_n, *) = \left(\mathcal{S}(A) \times \mathcal{S}(\bar{A}) \times \mathcal{S}(\bar{A}) \right) \cdot \left(\bigoplus_{i=1}^m \mathcal{S}(P_i) \right)$.

Proof. We show that for arbitrary $\sigma \in \text{Aut}(\mathcal{IS}_n, *)$ there exist $\tau = (g, h_1, h_2) \in (\mathcal{S}(A) \times \mathcal{S}(\bar{A}) \times \mathcal{S}(\bar{A}))$ and $\pi \in \left(\bigoplus_{i=1}^m \mathcal{S}(P_i) \right)$, such that $\sigma = \tau\pi$. For every $i \in A$ let $f_{\{i\}} = \begin{pmatrix} i \\ i \end{pmatrix}$. Proposition 1 implies that σ induces some permutation \tilde{g} on the set $\{f_{\{i\}}, i \in A\}$, which in its turn, induces the permutation of the indices $g \in \mathcal{S}(A)$.

Lemmas 6 and 7 imply that $\begin{pmatrix} j \\ j \end{pmatrix} \in \text{Ann}(\mathcal{IS}_n, *)$ for arbitrary $j \in \bar{A}$, moreover, using the equality $\sigma(\text{Ann}(\mathcal{IS}_n, *)) = \text{Ann}(\mathcal{IS}_n, *)$, we get $\sigma \begin{pmatrix} j \\ j \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$, where $s, t \in \bar{A}$.

Set $h_1(j) = s$, $h_2(j) = t$. Verify that the maps $h_1, h_2 : \bar{A} \rightarrow \bar{A}$ are bijections. Assume that $h_1(s) = h_1(l)$, $k+1 \leq s < l \leq n$. This means that

$$\sigma \begin{pmatrix} s \\ s \end{pmatrix} = \begin{pmatrix} h_1(s) \\ h_2(s) \end{pmatrix} \text{ and } \sigma \begin{pmatrix} l \\ l \end{pmatrix} = \begin{pmatrix} h_1(l) \\ h_2(l) \end{pmatrix}.$$

However in this case

$$\begin{pmatrix} s \\ s \end{pmatrix} = \sigma^{-1} \begin{pmatrix} h_1(s) \\ 1 \end{pmatrix} * \sigma^{-1} \begin{pmatrix} 1 \\ h_2(s) \end{pmatrix},$$

$$\begin{pmatrix} l \\ l \end{pmatrix} = \sigma^{-1} \begin{pmatrix} h_1(l) \\ 1 \end{pmatrix} * \sigma^{-1} \begin{pmatrix} 1 \\ h_2(l) \end{pmatrix},$$

hence, $s = l$ which contradicts the choice of s and l .

Now we prove that $\sigma \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} {}_g h_1(s) \\ {}_g h_2(t) \end{pmatrix}$, $s, t \in N$. Let $\sigma \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$. If $s, t \in A$ then $f_s * \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$. Taking the σ pre-images we get : $f_{g(s)} * \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$, hence $u = g(s) = {}_g h_1(s)$. In the same vein we get $\begin{pmatrix} u \\ v \end{pmatrix} * \varepsilon_{\{g(t)\}} = \begin{pmatrix} u \\ v \end{pmatrix}$ from the equality $\begin{pmatrix} s \\ t \end{pmatrix} * \varepsilon_{\{t\}} = \begin{pmatrix} s \\ t \end{pmatrix}$, and $v = g(t) = {}_g h_2(t)$.

Now let $s \in \bar{A}$, $t \in A$. Since $\begin{pmatrix} s \\ t \end{pmatrix} \in \text{Ann}_R(\mathcal{IS}_n, *)$ then $\sigma \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} u \\ g(t) \end{pmatrix} \in \text{Ann}_R(\mathcal{IS}_n, *)$, and $u \in \bar{A}$. Analogously since $\begin{pmatrix} t \\ s \end{pmatrix} \in \text{Ann}_L(\mathcal{IS}_n, *)$, then $\sigma \begin{pmatrix} t \\ s \end{pmatrix} = \begin{pmatrix} g(t) \\ v \end{pmatrix} \in \text{Ann}_L(\mathcal{IS}_n, *)$, and $v \in \bar{A}$. Applying σ to the both sides of the equality

$\begin{pmatrix} s \\ s \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} * \begin{pmatrix} t \\ s \end{pmatrix}$ we get $\begin{pmatrix} h_1(s) \\ h_2(s) \end{pmatrix} = \begin{pmatrix} u \\ g(t) \end{pmatrix} * \begin{pmatrix} g(t) \\ v \end{pmatrix}$. Hence $u = h_1(s)$, $v = h_2(s)$.

The case when $s \in A$, $t \in \bar{A}$ is treated in the same way.

Finally, let $s \in \bar{A}$, $t \in \bar{A}$. The equality

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \tau \begin{pmatrix} s \\ t \end{pmatrix} = \tau \left(\begin{pmatrix} s \\ 1 \end{pmatrix} * \begin{pmatrix} 1 \\ t \end{pmatrix} \right) = \\ &= \tau \begin{pmatrix} s \\ 1 \end{pmatrix} * \tau \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} h_1(s) \\ g(1) \end{pmatrix} * \begin{pmatrix} g(1) \\ h_2(t) \end{pmatrix} \end{aligned}$$

implies that $u = h_1(s)$, $v = h_2(t)$.

Let $a \in \mathcal{IS}_n$. We show that

$$M_1(\sigma(a)) = g(M_1(a)) \text{ and } (\sigma(a))(g(x)) = g(a(x)), \text{ for all } x \in M_1(a), \quad (1)$$

$$M_2(\sigma(a)) = g(M_2(a)) \text{ and } (\sigma(a))(g(x)) = h_2(a(x)), \text{ for all } x \in M_2(a), \quad (2)$$

$$M_3(\sigma(a)) = h_1(M_3(a)) \text{ and } (\sigma(a))(h_1(x)) = g(a(x)), \text{ for all } x \in M_3(a). \quad (3)$$

Prove, for example, (2), then statements (1) and (3) can be proved analogously.

Let $x \in M_2(a)$. Then $x \in A$, $a(x) \in \bar{A}$. From the equality $\begin{pmatrix} x \\ x \end{pmatrix} * a = \begin{pmatrix} x \\ a(x) \end{pmatrix}$, using the arguments from the previous paragraph, we get

$$\begin{pmatrix} g(x) \\ g(x) \end{pmatrix} * \sigma(a) = \sigma \left(\begin{pmatrix} x \\ x \end{pmatrix} * a \right) = \sigma \begin{pmatrix} x \\ a(x) \end{pmatrix} = \begin{pmatrix} g(x) \\ h_2(a(x)) \end{pmatrix}.$$

Therefore $g(x) \in \text{dom}(\sigma(a))$ and $(\sigma(a))(g(x)) = h_2(a(x))$. In particular, $g(x) \in M_2(\sigma(a))$. It follows that $g(M_2(a)) \subseteq M_2(\sigma(a))$ and $(\sigma(a))(g(x)) = h_2(a(x))$ for all $x \in M_2(a)$.

Conversely, let $x \in M_2(\sigma(a))$. Denote $y = (\sigma(a))(x)$. Then $x \in A$, $y \in \bar{A}$. Going to the σ^{-1} images in the equality $\begin{pmatrix} x \\ x \end{pmatrix} * \sigma(a) = \begin{pmatrix} x \\ y \end{pmatrix}$, we get $\begin{pmatrix} g^{-1}(x) \\ g^{-1}(x) \end{pmatrix} * a = \begin{pmatrix} g^{-1}(x) \\ h_2^{-1}(y) \end{pmatrix}$, hence $g^{-1}(x) \in \text{dom}(a)$ and $a(g^{-1}(x)) = h_2^{-1}(y)$, moreover, $h_2^{-1}(y) \in \bar{A}$. Then $g^{-1}(x) \in M_2(a)$, and $x \in g(M_2(a))$.

Now let $a = \begin{pmatrix} x_1 & \dots & x_l \\ y_1 & \dots & y_l \end{pmatrix} \in P$. Lemma 1 implies that $l \leq k$. Then $a = b * c$, where $b = \begin{pmatrix} x_1 & \dots & x_l \\ 1 & \dots & l \end{pmatrix}$, $c = \begin{pmatrix} 1 & \dots & l \\ y_1 & \dots & y_l \end{pmatrix}$. Therefore $\sigma(a) = \sigma(b) * \sigma(c)$, and hence

$$\text{dom}(\sigma(b)) = \{ {}_g h_1(x_1), \dots, {}_g h_1(x_l) \}, \quad \text{dom}(\sigma(c)) = \{ g(1), \dots, g(l) \}.$$

Moreover, for all $1 \leq i \leq l$ we have

$$\sigma(b)({}_g h_1(x_i)) = g(i), \quad \sigma(c)(g(i)) = {}_g h_2(y_i).$$

Hence,

$$\begin{aligned} \text{dom}(\sigma(a)) &= \{ {}_g h_1(x_1), \dots, {}_g h_1(x_l) \} \quad \text{and} \\ \sigma(a)({}_g h_1(x_i)) &= {}_g h_2(y_i), \quad 1 \leq i \leq l. \end{aligned}$$

It follows from (1), (2), (3) that

$$M_i(\sigma(a)) = M_i(\tau(a)) \text{ and } \sigma(a)(x) = \tau(a)(x)$$

for all $x \in M_i(\sigma(a))$, $1 \leq i \leq 3$. Hence $a \sim b$ if and only if $\sigma(a) \sim \sigma(b)$, that is if and only if $\tau(a) \sim \tau(b)$. So σ induces the permutation $\pi \in \bigoplus_{i=1}^m \mathcal{S}(P_i)$ such that $\sigma = \tau \cdot \pi$. \square

Lemma 9. $\bigoplus_{i=1}^m \mathcal{S}(P_i) \triangleleft \text{Aut}(\mathcal{IS}_n, *)$.

Proof. It is enough to prove that $\tau^{-1}\pi\tau \in \bigoplus_{i=1}^m \mathcal{S}(P_i)$ for arbitrary $\tau \in \mathcal{S}(A) \times \mathcal{S}(\bar{A}) \times \mathcal{S}(\bar{A})$, $\pi \in \bigoplus_{i=1}^m \mathcal{S}(P_i)$. we prove that $x \sim \tau^{-1}\pi\tau(x)$ for arbitrary $x \in N$. Indeed, $\tau^{-1}\pi(x) \sim \tau^{-1}(x)$. Hence $\tau^{-1}\pi\tau(x) \sim \tau^{-1}\tau(x) = x$. \square

The proof of theorem 2 follows immediately from the definition of the semidirect product, and Lemmas 5, 8 and 9. \square

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Department of Mechanics and Mathematics,
Kiev Taras Shevchenko University,
64, Volodymyrska st., 01033, Kiev, Ukraine
e-mail: G.K.: *akudr@univ.kiev.ua*

G.T.: *gtsyaputa@univ.kiev.ua*