

A presentation for the partial dual inverse symmetric monoid

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Abstract

We give a monoid presentation in terms of generators and defining relations for the partial analogue of the finite dual inverse symmetric monoid.

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1 Introduction

The partial dual inverse symmetric monoid on a set X , denoted by \mathcal{PIP}_X , is a partial analogue of the dual inverse symmetric monoid \mathcal{IP}_X , see [M] and [FL]. This monoid is a natural generalization of the full inverse symmetric monoid and has a number of interesting properties, which were studied in [KMal].

The aim of the present paper is to obtain a presentation for the monoid \mathcal{PIP}_X with X finite in terms of generators and defining relations. We would like to mention that during the recent period there appeared a number of papers where presentations for some important transformation semigroups and their generalizations (e.g., so called Brauer-type semigroups) have been found, see [Fer], [F], [E], [KMaz], [MM]. In view of this, our research looks like a natural continuation of the previous efforts.

It is interesting that by the moment (so far as to our knowledge) a presentation for the finite dual inverse symmetric monoid \mathcal{IP}_X is not found. Some possible approaches towards finding such a presentation and the arising difficulties are discussed in [EEF]. In view of this, our result looks somehow unexpected, as we solve the problem for a bigger and more complicated monoid. The authors have a hope that the ideas and technique suggested in the present paper could be utilized, in particular, for finding a presentation for the monoid \mathcal{IP}_X .

The paper is organized as follows. In Section 2 we recall the definition of the monoid \mathcal{PIP}_X (and of the monoid \mathcal{IP}_X). In Section 3 we define an abstract monoid S by generators and defining relations and establish some other relations which are the consequences of the defining ones. Further, in Section 4 we continue investigating the monoid S and develop some rewriting technique for the elements of S presented as words over its generators. Using this technique we manage to show that every element of S can be presented as a certain "canonical word". Finally, in Section 5 we turn back to the monoid \mathcal{PIP}_X , X finite, construct a natural epimorphism from S onto \mathcal{PIP}_X and show that this epimorphism is in fact an isomorphism. For this, we prove that the presentation of an element of \mathcal{PIP}_X as an image of some canonical word is unique, and thus the cardinality of S does not exceed the cardinality of \mathcal{PIP}_X .

2 Definition of the monoid \mathcal{PIP}_n

Let X be a set. Consider a set $X' = \{x'\}_{x \in X}$ disjoint with X and a bijection $' : X \rightarrow X'$ sending $x \in X$ to $x' \in X'$. Denote the inverse bijection by the same symbol, that is $(x')' = x$ for all $x \in X \cup X'$.

We shall say that a subset A of $X \cup X'$ is a

- *line* provided that $A \cap X \neq \emptyset$ and $A \cap X' \neq \emptyset$;
- *point* provided that $|A| = 1$.

Let \mathcal{IP}_X be the set of all decompositions of $X \cup X'$ into lines, and \mathcal{PIP}_X the set of all decompositions of $X \cup X'$ into lines and points. Obviously, $\mathcal{IP}_X \subset \mathcal{PIP}_X$.

In the case when $X = \{1, \dots, n\}$ we shall denote \mathcal{IP}_X by \mathcal{IP}_n and \mathcal{PIP}_X by \mathcal{PIP}_n .

Let $a \in \mathcal{PIP}_X$ and $x, y \in X \cup X'$. Set $x \equiv_a y$ provided that x and y are of the same block of a . The map $a \mapsto \equiv_a$ is a bijection between the elements of \mathcal{PIP}_X and the equivalence relations on $X \cup X'$ whose classes are either points or lines. Under this bijection the set \mathcal{IP}_X maps onto the set of those equivalence relations on $X \cup X'$ whose classes are lines.

To define a multiplication on the set \mathcal{IP}_X we consider any $a, b \in \mathcal{IP}_X$ and define a new equivalence relation, \equiv , on $X \cup X'$ as follows:

- for $x, y \in X$ we have $x \equiv y$ if and only if $x \equiv_a y$ or there is a sequence, c_1, \dots, c_{2s} , $s \geq 1$, of elements of X , such that $x \equiv_a c'_1$, $c_1 \equiv_b c_2$, $c'_2 \equiv_a c'_3$, \dots , $c_{2s-1} \equiv_b c_{2s}$, and $c'_{2s} \equiv_a y$;

- for $x, y \in X$ we have $x' \equiv y'$ if and only if $x' \equiv_b y'$ or there is a sequence, c_1, \dots, c_{2s} , $s \geq 1$, of elements of X , such that $x' \equiv_b c_1$, $c'_1 \equiv_a c'_2$, $c_2 \equiv_b c_3, \dots, c'_{2s-1} \equiv_a c'_{2s}$, and $c_{2s} \equiv_b y'$;
- for $x, y \in X$ we have $x \equiv y'$ if and only if $y' \equiv x$ if and only if there is a sequence, c_1, \dots, c_{2s-1} , $s \geq 1$, of elements of X , such that $x \equiv_a c'_1$, $c_1 \equiv_b c_2$, $c'_2 \equiv_a c'_3, \dots, c'_{2s-2} \equiv_a c'_{2s-1}$, and $c_{2s-1} \equiv_b y'$.

Since every class of \equiv is a line this definition is correct. We set the decomposition of $X \cup X'$ into \equiv -classes to be the product $a \cdot b$ of a and b in \mathcal{IP}_X . With respect to this multiplication (\mathcal{IP}_X, \cdot) is a semigroup. It was called the *inverse partition semigroup* on the set X in [M] and [M1], the *monoid of block bijections* in [EEF], and the *dual inverse symmetric monoid* on the set X in [FL] and a number of subsequent papers. In this paper we stick to the latter term.

Let $x \notin X$ be an arbitrary element. Set $Y = X \cup \{x\}$ and denote by $\widetilde{\mathcal{IP}}_Y$ the subset of \mathcal{IP}_Y consisting of those decomposition of $Y \cup Y'$ into subsets which consist entirely of lines and both x and x' belong to the same line. The set $\widetilde{\mathcal{IP}}_Y$ is closed with respect to the operation \cdot and is therefore a subsemigroup of \mathcal{IP}_Y .

Take $a \in \mathcal{PIP}_X$ and denote by $\varphi(a)$ the element of $\widetilde{\mathcal{IP}}_Y$, consisting of all lines of a and of one additional block, whose elements are x, x' and all points of a . It was noticed in [KMal] (and is easy to see) that the map φ is a bijection from the set \mathcal{PIP}_X onto the set $\widetilde{\mathcal{IP}}_Y$. Now we are prepared to define the (associative) multiplication on \mathcal{PIP}_X . We set (slightly abusing the notation)

$$a \cdot b = \varphi^{-1}(\varphi(a) \cdot \varphi(b)).$$

The above defined multiplication in the monoid \mathcal{PIP}_X has a natural realization as a "superposition of diagrams". We interpret the elements of \mathcal{PIP}_X as diagrams with vertices on the left hand side indexed by X and vertices on the right hand side indexed by X' . To multiply two such diagrams α and β one has to place β to the right of α such that the corresponding right vertices of α and left vertices of β are identified, which uniquely determines the diagram of the product decomposition $\alpha\beta$. This is illustrated on Figures 1 and 2.

The semigroup (\mathcal{PIP}_X, \cdot) is a "partial analogue" of the semigroup \mathcal{IP}_X . In particular, it contains the semigroup (\mathcal{IP}_X, \cdot) as a subsemigroup. The structure of the semigroup (\mathcal{PIP}_X, \cdot) was investigated in [KMal], where it was called the *partial inverse partition semigroup*. We would like, developing the terminology stemming from [FL], to propose a more apt, from our point

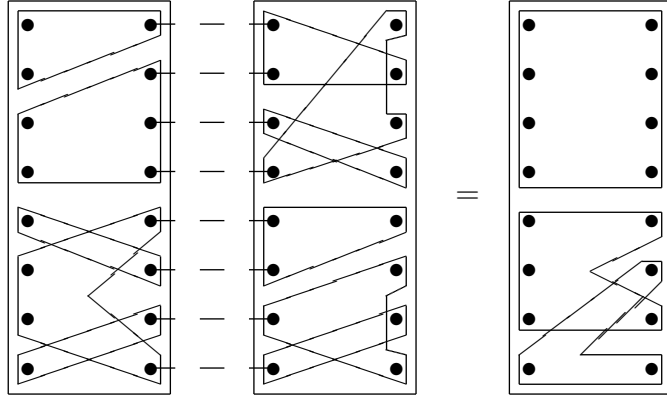


Figure 1: Elements of \mathcal{IP}_8 and their multiplication.

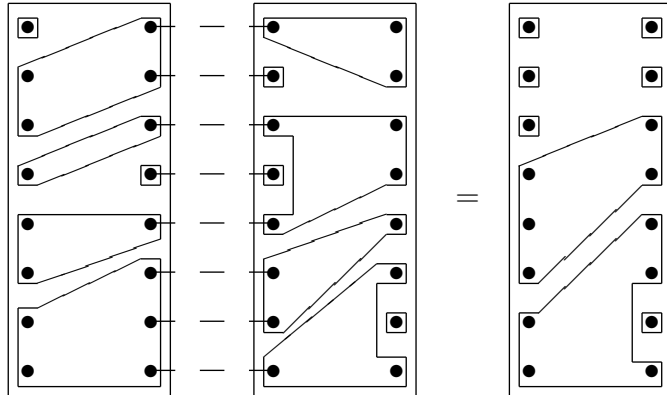


Figure 2: Elements of \mathcal{PIP}_8 and their multiplication.

of view, term for the monoid \mathcal{PIP}_X , the *partial dual inverse symmetric monoid*.

3 The abstract monoid S and its relations

Let $n \geq 3$. Consider the monoid S with the identity element e generated by $\sigma_1, \dots, \sigma_{n-1}; \lambda_1, \dots, \lambda_{n-1}; \rho_1, \dots, \rho_{n-1}; e_1, \dots, e_n$ subject to the following

relations:

$$\sigma_i^2 = e, 1 \leq i \leq n-1, \quad (1)$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| > 1, \quad (2)$$

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, |i-j| = 1, \quad (3)$$

$$\lambda_i \lambda_j = \lambda_j \lambda_i, \rho_i \rho_j = \rho_j \rho_i, \rho_i \lambda_j = \lambda_j \rho_i, |i-j| > 1, \quad (4)$$

$$\lambda_i \rho_{i+1} = \sigma_{i+1} \rho_i \lambda_i e_{i+2} = \sigma_i \rho_{i+1} \sigma_i \lambda_i, \lambda_{i+1} \rho_i = e_{i+2} \rho_i \lambda_i \sigma_{i+1} = \rho_i \sigma_i \lambda_{i+1} \sigma_i, \quad (5)$$

$$\rho_i \lambda_{i+1} = \rho_i e_{i+2}, \rho_{i+1} \lambda_i = e_{i+2} \lambda_i, \quad (6)$$

$$\lambda_i \rho_i \lambda_i = \lambda_i, \rho_i \lambda_i \rho_i = \rho_i, \quad (7)$$

$$\sigma_i \lambda_j \sigma_i = \sigma_j \lambda_i \sigma_j, \sigma_i \rho_j \sigma_i = \sigma_j \rho_i \sigma_j, |i-j| = 1, \quad (8)$$

$$\lambda_i \sigma_i = \lambda_i, \sigma_i \rho_i = \rho_i, 1 \leq i \leq n-1, \quad (9)$$

$$\sigma_i \lambda_j = \lambda_j \sigma_i, \sigma_i \rho_j = \rho_j \sigma_i, |j-i| > 1, \quad (10)$$

$$\lambda_i \lambda_{i+1} = \lambda_i \lambda_{i+1} \sigma_i = \sigma_{i+1} \lambda_i \lambda_{i+1}, \rho_{i+1} \rho_i = \sigma_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \sigma_{i+1}, \quad (11)$$

$$\sigma_{i+1} \lambda_{i+1} \lambda_i = \lambda_{i+1} \lambda_i = \lambda_i e_{i+2}, \rho_i \rho_{i+1} \sigma_{i+1} = \rho_i \rho_{i+1} = e_{i+2} \rho_i, \quad (12)$$

$$e_i = \lambda_{i-1} \rho_{i-1}, i \geq 2, e_1 = \sigma_1 e_2 \sigma_1, \quad (13)$$

$$e_i^2 = e_i, e_i e_{i+1} = e_{i+1} e_i = \lambda_i^2 = \rho_i^2 = \rho_i \sigma_i \lambda_i, \quad (14)$$

$$e_i \sigma_j = \sigma_j e_i, j \neq i, i-1, e_i \sigma_i = \sigma_i e_{i+1}, \sigma_i e_i = e_{i+1} \sigma_i, \quad (15)$$

$$e_i \lambda_j = \lambda_j e_i, j \neq i, i-1, e_{i+1} \lambda_i = \lambda_i, e_i \lambda_i = \lambda_i e_{i+1} = \lambda_i e_i = e_i e_{i+1}, \quad (16)$$

$$e_i \rho_j = \rho_j e_i, j \neq i, i-1, \rho_i e_{i+1} = \rho_i, \rho_i e_i = e_{i+1} \rho_i = e_i \rho_i = e_i e_{i+1}. \quad (17)$$

Remark 1. It follows from (13) that S is generated by λ_i -s, ρ_i -s and σ_i -s only since the defining relations can be readily rewritten without e_i -s. However, it is convenient for us to include e_i -s to the generating set and to the relations, because the products of e_i -s will appear afterwards in the canonical words we will introduce.

Remark 2. We would like to emphasize that the proposed set of relations does not pretend to being irreducible. For example, the relations $e_i^2 = e_i$ from (14) follow from the relations (7). However, we keep these and some other redundant relations with the purpose of making the subsequent text more transparent and readable.

In view of the relations (1), (2), (3) the submonoid of S , generated by all σ_i -s, is isomorphic to the full symmetric group \mathcal{S}_n . From now on, identify this submonoid with \mathcal{S}_n .

Let $\pi \in \mathcal{S}_n$ and $\alpha \in S$. Set $\alpha^\pi = \pi^{-1} \alpha \pi$. Obviously, the map $\varphi_\pi : \alpha \mapsto \alpha^\pi$ is an automorphism of S , and $\pi \mapsto \varphi_\pi$ is an action of \mathcal{S}_n on S . Call this action *the action by inner automorphisms*.

Let $1 \leq i < j \leq n$. Set

$$\sigma_{i,j} = \begin{cases} \sigma_i, & \text{if } j = i + 1; \\ \sigma_i \sigma_{i+1} \dots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2} \dots \sigma_i, & \text{if } j > i + 1. \end{cases}$$

For $1 \leq j < i \leq n$ we set $\sigma_{j,i} = \sigma_{i,j}$. Notice that $\sigma_{i,j}^2 = e$ for all acceptable i, j .

Lemma 1. *Let $1 \leq i \leq n - 1$. Consider the action of \mathcal{S}_n on S by inner automorphisms.*

1. *If $2 \leq i \leq n - 2$, then the elements $\sigma_1, \dots, \sigma_{i-2}, \sigma_{i+2}, \dots, \sigma_n$ and $\sigma_{i-1, i+2}$ stabilize both λ_i and ρ_i .*
2. *The elements $\sigma_3, \dots, \sigma_n$ stabilize both λ_1 and ρ_1 , the elements $\sigma_1, \dots, \sigma_{n-2}$ stabilize both λ_n and ρ_n .*
3. *The elements $\sigma_1, \dots, \sigma_{i-2}, \sigma_{i+1}, \dots, \sigma_n$ and $\sigma_{i-1, i+1}$ stabilize e_i , $2 \leq i \leq n - 1$.*
4. *The elements $\sigma_2, \dots, \sigma_{n-1}$ stabilize e_1 , the elements $\sigma_1, \dots, \sigma_{n-2}$ stabilize e_n .*

Proof. To prove the first claim, in view of (10), we have only to show that $\sigma_{i-1, i+2}$ stabilizes λ_i and ρ_i , $2 \leq i \leq n - 2$. Applying subsequently (8), (1), (10), (1), (8), (1), we obtain

$$\begin{aligned} \sigma_{i-1, i+2} \lambda_i \sigma_{i-1, i+2} &= \sigma_{i-1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i-1} \lambda_i \sigma_{i-1} \sigma_i \sigma_{i+1} \sigma_i \sigma_{i-1} = \\ &= \sigma_{i-1} \sigma_i \sigma_{i+1} \lambda_{i-1} \sigma_{i+1} \sigma_i \sigma_{i-1} = \sigma_{i-1} \sigma_i \lambda_{i-1} \sigma_i \sigma_{i-1} = \lambda_i, \end{aligned}$$

as required. For ρ_i arguments are similar.

To prove the third claim we let $2 \leq i \leq n - 1$ and show that $\sigma_{i-1, i+1}$ stabilizes e_i . Indeed, using (15) we compute

$$\begin{aligned} \sigma_{i-1} \sigma_i \sigma_{i-1} e_i \sigma_{i-1} \sigma_i \sigma_{i-1} &= \sigma_{i-1} \sigma_i e_{i-1} \sigma_i \sigma_{i-1} = \sigma_{i-1} e_{i-1} \sigma_i \sigma_{i-1} = \\ &= \sigma_{i-1} e_{i-1} \sigma_{i-1} = e_i, \end{aligned}$$

as required.

The remaining two claims are proved similarly, and we leave the details to the reader. \square

To proceed, we need to introduce some more notation. Let $1 \leq p, q \leq n$ and $p \neq q$. For any $\pi \in \mathcal{S}_n$ such that $\pi(1) = p$ and $\pi(2) = q$ set

$$\lambda_{p,q} = \pi^{-1} \lambda_1 \pi, \quad \rho_{p,q} = \pi^{-1} \rho_1 \pi. \quad (18)$$

In view of Lemma 1 this definition is correct, i.e. independent on the choice of $\pi \in \mathcal{S}_n$ such that $\pi(1) = p$ and $\pi(2) = q$. Moreover, it can be easily verified that

$$\lambda_{i,i+1} = \lambda_i, \quad \rho_{i,i+1} = \rho_i$$

for all $1 \leq i \leq n-1$. Indeed, for $i = 1$ this is trivial. Let $i \geq 2$. Then we apply (8) and (1) $(i-1)$ times and obtain

$$(\sigma_{i-1}\sigma_i) \cdots (\sigma_2\sigma_3)(\sigma_1\sigma_2)\lambda_1(\sigma_2\sigma_1)(\sigma_3\sigma_2) \cdots (\sigma_i\sigma_{i-1}) = \lambda_i.$$

Besides, the element $(\sigma_2\sigma_1)(\sigma_3\sigma_2) \cdots (\sigma_i\sigma_{i-1})$ maps 1 to i and 2 to $i+1$ respectively.

Lemma 2. *Let $\pi \in \mathcal{S}_n$ be such that $\pi(p) = s$ and $\pi(q) = t$. Then $\pi^{-1}\lambda_{p,q}\pi = \lambda_{s,t}$ and $\pi^{-1}\rho_{p,q}\pi = \rho_{s,t}$.*

Proof. We prove only the first equality, the second one being proved similarly. Firstly, we note that every element $\alpha \in \mathcal{S}_n$ such that $\alpha(s) = s$ and $\alpha(t) = t$ stabilizes $\lambda_{s,t}$. This follows from the definition of $\lambda_{s,t}$ and Lemma 1. Further, consider γ and δ from \mathcal{S}_n such that $\gamma(1) = s$, $\gamma(2) = t$, $\delta(1) = p$, $\delta(2) = q$. Then

$$\delta^{-1}\gamma\lambda_{p,q}\gamma^{-1}\delta = \delta^{-1}\lambda_1\delta = \lambda_{s,t},$$

which yields the required statement. \square

Lemma 3. *Let $\pi \in \mathcal{S}_n$ be such that $\pi(p) = s$. Then $\pi^{-1}e_p\pi = e_s$.*

Proof. Let $\alpha \in \mathcal{S}_n$ be such that $\alpha(1) = p$. Then $\alpha^{-1}e_1\alpha = e_p$ by (13), (15) and Lemma 1. Then we apply the arguments similar to those from the proof of the previous lemma. \square

In the following proposition we collect the relations satisfied by the products of elements $\lambda_{p,q}$, $\rho_{p,q}$, $\sigma_{p,q}$ by e_i .

Proposition 1. *The following relations hold for all admissible and pairwise distinct p, q, k :*

$$e_p^2 = e_p, \quad e_p e_q = e_q e_p, \tag{19}$$

$$e_k \sigma_{p,q} = \sigma_{p,q} e_k, \quad e_p \sigma_{p,q} = \sigma_{p,q} e_q, \tag{20}$$

$$e_k \lambda_{p,q} = \lambda_{p,q} e_k, \quad e_q \lambda_{p,q} = \lambda_{p,q}, \quad e_p \lambda_{p,q} = \lambda_{p,q} e_q = \lambda_{p,q} e_p = e_p e_q \tag{21}$$

$$e_k \rho_{p,q} = \rho_{p,q} e_k, \quad \rho_{p,q} e_q = \rho_{p,q}, \quad \rho_{p,q} e_p = e_q \rho_{p,q} = e_p \rho_{p,q} = e_p e_q \tag{22}$$

Proof. Consider the map $' : S \rightarrow S$ defined by $\sigma'_i = \sigma_i$, $\lambda'_i = \rho_i$, $\rho'_i = \lambda_i$, $e'_i = e_i$. In view of the defining relations this map uniquely extends to an involution on S which we will also denote by $'$.

The relations (19) follow from (14) and Lemma 3.

The relations (20) follow from (15) and Lemma 3.

The relations (21) follow from (16), applying Lemma 3.

Finally, (22) follows from (21), using $'$. \square

Proposition 2. *For all pairwise distinct p, q, k, l ,*

$$\lambda_{p,q}^2 = \rho_{p,q}^2 = \lambda_{p,q}\lambda_{q,p} = \rho_{p,q}\rho_{q,p} = e_p e_q, \quad (23)$$

$$\lambda_{k,l}\lambda_{p,q} = \lambda_{p,q}\lambda_{k,l}, \quad \rho_{k,l}\rho_{p,q} = \rho_{p,q}\rho_{k,l}, \quad \lambda_{k,l}\rho_{p,q} = \rho_{p,q}\lambda_{k,l}, \quad (24)$$

$$\lambda_{k,q}\lambda_{k,l} = \lambda_{k,l}\lambda_{k,q} = \lambda_{k,q}\lambda_{q,l}, \quad \rho_{k,l}\rho_{k,q} = \rho_{k,q}\rho_{k,l} = \rho_{l,q}\rho_{k,l}, \quad (25)$$

$$\lambda_{k,l}\lambda_{p,k} = e_l\lambda_{p,k}, \quad \rho_{p,k}\rho_{k,l} = e_l\rho_{p,k}, \quad (26)$$

$$\lambda_{k,l}\lambda_{p,l} = e_k\lambda_{p,l}, \quad \rho_{p,l}\rho_{k,l} = e_k\rho_{p,l}, \quad (27)$$

$$\lambda_{k,l}\rho_{l,k} = e_l\sigma_{k,l}, \quad \lambda_{k,l}\rho_{k,l} = e_l, \quad \rho_{k,l}\lambda_{l,k} = e_k e_l, \quad (28)$$

$$\lambda_{k,l}\rho_{k,q} = \rho_{k,q}\lambda_{k,l} = \rho_{k,q}\lambda_{k,q}e_l\sigma_{q,l}, \quad \rho_{k,l}\lambda_{p,k} = \lambda_{p,k}e_l, \quad (29)$$

$$\lambda_{k,l}\rho_{p,k} = \rho_{p,k}\lambda_{p,k}e_l\sigma_{k,l}, \quad \lambda_{p,k}\rho_{k,l} = \sigma_{k,l}e_l\rho_{p,k}\lambda_{p,k} \quad \rho_{p,k}\lambda_{k,l} = e_l\rho_{p,k}, \quad (30)$$

$$\lambda_{k,l}\rho_{p,l} = \sigma_{p,l}e_p\rho_{k,l}\lambda_{k,l}\sigma_{p,l}, \quad (31)$$

$$\rho_{k,l}\lambda_{k,l} = \rho_{l,k}\lambda_{l,k}. \quad (32)$$

Proof. To prove (23), in view of (14) and applying Lemmas 3, 2 and $'$, it is enough to check that $\lambda_1\lambda_{2,1} = e_1e_2$. Indeed, using (9) and (14), we have

$$\lambda_1\lambda_{2,1} = \lambda_1\sigma_1\lambda_1\sigma_1 = \lambda_1^2 = e_1e_2.$$

The relations (24) follow from (4) and Lemma 2.

To prove (25), in view of Lemma 2 and applying $'$, it is enough to prove that $\lambda_1\lambda_{1,3} = \lambda_{1,3}\lambda_1 = \lambda_1\lambda_2$. Indeed, applying (9), (11) and then (11), (9), (11), (9), (8), (8) we compute

$$\lambda_1\lambda_{1,3} = \lambda_1\sigma_1\lambda_2\sigma_1 = \lambda_1\lambda_2\sigma_1 = \lambda_1\lambda_2;$$

$$\begin{aligned} \lambda_1\lambda_2 &= \sigma_2\lambda_1\lambda_2 = \sigma_2\lambda_1\lambda_2\sigma_2 = \sigma_2\lambda_1\lambda_2\sigma_1\sigma_2 = \sigma_2\lambda_1\sigma_1\lambda_2\sigma_1\sigma_2 = \\ &= \sigma_2\lambda_1\sigma_2\lambda_1\sigma_2\sigma_2 = \sigma_1\lambda_2\sigma_1\lambda_1 = \lambda_{1,3}\lambda_1. \end{aligned}$$

To prove (26) it is enough to show that $\lambda_2\lambda_1 = \lambda_1e_3$, which holds by (12).

To prove (27) it is enough to check that $\lambda_2\lambda_{1,3} = e_2\lambda_{1,3}$. Conjugating both sides with σ_2 we obtain the equivalent equality $\sigma_2\lambda_2\lambda_1 = e_3\lambda_1$, which holds by (12).

The first equality of (28) follows from $\lambda_2\rho_{3,2} = \lambda_2\rho_2\sigma_2 = e_3\sigma_2$ and Lemmas 2 and 3. The second and the third equalities follow from the same lemmas in view of (14).

In order to prove the first relation of (29) it is again enough to verify the relation $\rho_{1,3}\lambda_1 = \lambda_1\rho_{1,3} = \rho_{1,3}\lambda_{1,3}e_2\sigma_2$. We compute

$$\begin{aligned}\lambda_1\rho_{1,3} &= \lambda_1\sigma_1\rho_2\sigma_1 = \lambda_1\rho_2\sigma_1 = \sigma_2\rho_1\lambda_1e_3\sigma_1 = \sigma_2\rho_1\lambda_1e_3 = \sigma_2\rho_1\lambda_1\sigma_2e_2\sigma_2 = \\ &\quad \sigma_2\rho_1\sigma_2\sigma_2\lambda_1\sigma_2e_2\sigma_2 = \sigma_1\rho_2\sigma_1\sigma_1\lambda_2\sigma_1e_2\sigma_2 = \rho_{1,3}\lambda_{1,3}e_2\sigma_2\end{aligned}$$

and

$$\lambda_1\rho_{1,3} = \lambda_1\rho_2\sigma_1 = \sigma_1\rho_2\sigma_1\lambda_1\sigma_1 = \rho_{1,3}\lambda_1\sigma_1 = \rho_{1,3}\lambda_1.$$

To prove the second relation of (29) it suffices to establish only that $\rho_2\lambda_1 = \lambda_1e_3$, which is a direct consequence of (6). Further, applying ' to this relation we obtain the third relation of (29).

To prove the first relation of (30) we verify only that $\lambda_2\rho_1 = \rho_1\lambda_1e_3\sigma_2$, which is a direct consequence of (5). The second relation of (30) is a consequence of the first one using '.

Further, let us prove (31). It is enough to establish that $\lambda_1\rho_{3,2} = \sigma_2e_3\rho_1\lambda_1\sigma_2$. This equality is equivalent to $\lambda_1\rho_2 = \sigma_2e_3\rho_1\lambda_1$, which, in turn, follows from (5).

Finally, the relation (32) follows from $\sigma_1\rho_1\sigma_1\sigma_1\lambda_1\sigma_1 = \rho_1\lambda_1$ and Lemma 2. The proof is complete. \square

4 Rewriting technique and canonical words in the monoid S

In this section we are going to develop some rewriting technique in order to show that any element of the monoid S can be represented by some "reduced word". This will imply that the cardinality of S is not bigger than the cardinality of the set of all reduced words.

We start from the following observation.

Lemma 4. *Every element of S can be written as a product of the form $\alpha_1 \dots \alpha_k \beta$, where $k \geq 0$, each α_i is equal to some $\lambda_{p,q}$ or $\rho_{p,q}$ and $\beta \in \mathcal{S}_n$.*

Proof. Let $\alpha \in S$. Since every λ_i and ρ_i belongs to some orbit of λ_1 and ρ_1 respectively with respect to the action of \mathcal{S}_n by inner automorphisms, S can be generated by \mathcal{S}_n , λ_1 and ρ_1 . Therefore, α can be written in the form

$$\alpha = \pi_1\gamma_1\pi_2\gamma_2 \dots \pi_k\gamma_k\pi_{k+1},$$

where $k \geq 0$, $\pi_i \in \mathcal{S}_n$ for all i and each γ_i equals either λ_1 or ρ_1 . In view of (18) we can rewrite the expression for α as follows:

$$\alpha = \pi_1 \gamma_1 \pi_1^{-1} (\pi_1 \pi_2) \gamma_2 (\pi_1 \pi_2)^{-1} \dots (\pi_1 \dots \pi_k) \gamma_k (\pi_1 \dots \pi_k)^{-1} \cdot (\pi_1 \dots \pi_k \pi_{k+1}) = \gamma_{p_1, q_1}^1 \dots \gamma_{p_k, q_k}^k \beta,$$

where $p_i = (\pi_1 \dots \pi_i)^{-1}(1)$, $q_i = (\pi_1 \dots \pi_i)^{-1}(2)$ and

$$\gamma_{p_i, q_i}^i = \begin{cases} \lambda_{p_i, q_i}, & \text{if } \gamma_i = \lambda_1, \\ \rho_{p_i, q_i}, & \text{if } \gamma_i = \rho_1, \end{cases}$$

$$1 \leq i \leq k, \text{ and } \beta = \pi_1 \dots \pi_{k+1}. \quad \square$$

Lemma 5. *Every element of S can be written as a product of the form*

$$\alpha_1 \dots \alpha_k \beta_1 \dots \beta_l \pi E, \quad (33)$$

where $k, l \geq 0$, each α_i equals some $\rho_{p,q}$, each β_i equals some $\lambda_{p,q}$, $\pi \in \mathcal{S}_n$ and $E = e$ or E is a product of several e_p -s.

Proof. Let $\alpha \in S$. It follows from Lemma 4 that we can express α as a product $\alpha_1 \dots \alpha_k \pi$, where $k \geq 0$, each α_i is equal to some $\lambda_{p,q}$ or $\rho_{p,q}$ and $\pi \in \mathcal{S}_n$. Suppose that $\alpha_i = \lambda_{p,q}$ and $\alpha_{i+1} = \rho_{k,l}$ for some i . If the sets $\{p, q\}$ and $\{k, l\}$ are disjoint we have $\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i$ by (24). If the sets $\{p, q\}$ and $\{k, l\}$ are not disjoint we apply the appropriate relation of (28)-(31). As a result we obtain an expression for α containing less subwords of the form $\lambda_{i,j} \rho_{s,t}$.

However, after such a rewriting some e_i -s and $\sigma_{s,t}$ -s might appear in the expression for α . If some e_i -s appear, using (20), (21) and (22) we rewrite our expression such that it has the occurrence of e_j at the rightmost position, while the number of subwords of the form $\lambda_{i,j} \rho_{s,t}$ remains the same. If some $\sigma_{s,t}$ -s appear, using the action of \mathcal{S}_n on S by inner automorphisms, we can, similarly to as this is done in the proof of Lemma 4, rewrite it such that the group element occurs to the right to all occurrences of $\lambda_{i,j}$ -s and $\rho_{s,t}$ -s. As the mentioned rewriting does not affect the number of the subwords of the form $\lambda_{i,j} \rho_{s,t}$, the statement of the lemma follows by induction on the number of subwords of the form $\lambda_{i,j} \rho_{s,t}$ in the initial expression for α . \square

We can even strengthen the previous statement.

Lemma 6. *Every element of S can be written as a product of the form (33) such that the conditions of Lemma 5 are satisfied and, moreover, the following conditions are also satisfied:*

1. If $\alpha_i = \rho_{p,q}$ and $\alpha_j = \rho_{k,l}$ then either $\{p, q\} \cap \{k, l\} = \emptyset$ or $\{p, q\} \cap \{k, l\} = \{p\} = \{k\}$, so that $\alpha_i = \rho_{p,q}$ and $\alpha_j = \rho_{p,l}$. In particular, α_i and α_j commute.
2. If $\beta_i = \lambda_{p,q}$ and $\beta_j = \lambda_{k,l}$ then either $\{p, q\} \cap \{k, l\} = \emptyset$ or $\{p, q\} \cap \{k, l\} = \{p\} = \{k\}$, so that $\beta_i = \lambda_{p,q}$ and $\beta_j = \lambda_{p,l}$. In particular, β_i and β_j commute.

Proof. We will prove the statement on α_i -s only, the second statement being proved analogously.

Notice that it is enough to prove the statement for the case $j = i + 1$. Indeed, if α_i commutes with α_{i+1} for every i then we can rearrange the factors of α such that α_j follows α_i for any i, j .

Apply induction on the number of factors of the form $\rho_{p,q}$ in the expression (33). If this number is zero or one, the statement is obvious. Suppose that $\alpha_1 = \rho_{p,q}$ and $\alpha_2 = \rho_{k,l}$, where p, q, l are pairwise distinct, and that $\{p, q\} \cap \{k, l\} \neq \emptyset$. Consider six possible cases:

- a) if $\alpha_i = \rho_{p,q}, \alpha_{i+1} = \rho_{p,q}$, we apply (23),
- b) if $\alpha_i = \rho_{p,q}, \alpha_{i+1} = \rho_{q,p}$, we apply (23),
- c) if $\alpha_i = \rho_{p,q}, \alpha_{i+1} = \rho_{p,l}$, then $\alpha_i \alpha_{i+1} = \rho_{p,q} \rho_{p,l} = \rho_{p,l} \rho_{p,q}$ by (25),
- d) if $\alpha_i = \rho_{p,q}, \alpha_{i+1} = \rho_{l,p}$, then the product $\alpha_i \alpha_{i+1}$ equals the product from c) by (25),
- e) if $\alpha_i = \rho_{p,q}, \alpha_{i+1} = \rho_{l,q}$, we apply (27),
- f) if $\alpha_i = \rho_{p,q}, \alpha_{i+1} = \rho_{q,l}$, we apply (26).

In the cases a), b), e), f) we obtain an expression for the initial element containing less entries of factors of the form $\rho_{p,q}$ and apply the inductive hypothesis. In the cases c) and d) we have that $\alpha_i \alpha_{i+1} = \rho_{p,q} \rho_{p,k}$ for some pairwise distinct p, q, k .

We proceed by considering the product $\alpha_2 \alpha_3$ and so on. Finally we either reach the last factor α_l with the first claim satisfied for every possible α_i and α_{i+1} , or reduce the number of factors. In the latest case we apply the induction. \square

Let $p \in \{1, \dots, n\}$, $A \subseteq \{1, \dots, n\}$ and $p \notin A$. Set $R_{p,A} = \rho_{p,a_1} \cdots \rho_{p,a_s}$ and $L_{p,A} = \lambda_{p,a_1} \cdots \lambda_{p,a_s}$, where $A = \{a_1, \dots, a_s\}$. In view of (25) $R_{p,A}$ and $L_{p,A}$ are well-defined.

Corollary 1. *Suppose $A_1 \cap A_2 = \emptyset$, $p_1 \neq p_2$, $p_1 \notin A_1$, $p_2 \notin A_2$. Then $R_{p_1, A_1} R_{p_2, A_2} = R_{p_2, A_2} R_{p_1, A_1}$ and $L_{p_1, A_1} L_{p_2, A_2} = L_{p_2, A_2} L_{p_1, A_1}$.*

For a subset $M = \{m_1, \dots, m_s\} \subset \{1, \dots, n\}$ set $E_M = e_{m_1} \cdots e_{m_s}$, which is well-defined in view of (19).

Proposition 3. *Every element of S can be written as a product of the form*

$$R_{p_1, A_1} \cdots R_{p_k, A_k} L_{q_1, B_1} \cdots L_{q_l, B_l} E_M \sigma, \quad (34)$$

where $k, l \geq 0$, $p_1, \dots, p_k, q_1, \dots, q_l$ are pairwise distinct and $A_1, \dots, A_k, B_1, \dots, B_l$ are pairwise disjoint, $E = E_M$, $M \subseteq \{1, \dots, n\}$, $\sigma \in \mathcal{S}_n$. Moreover, the following conditions are satisfied:

- (i) $p_i \notin B_1 \cup \cdots \cup B_l$, $1 \leq i \leq k$,
- (ii) $q_i \notin A_1 \cup \cdots \cup A_k$, $1 \leq i \leq l$,
- (iii) M is disjoint with $(\cup_{i=1}^k (\{p_i\} \cup A_i)) \cup (\cup_{i=1}^l (\{q_i\} \cup B_i))$.

Proof. Let $\alpha \in S$ be presented in the form (33), such that the conditions of Lemma 6 are satisfied. The relations (20) imply that we can move from the expression (33) to

$$\alpha = \alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_l E' \pi', \quad (35)$$

where E' is the product of some e_i -s and $\pi' \in \mathcal{S}_n$. It follows from Lemma 6 that we can rearrange the factors in the product $\alpha_1 \cdots \alpha_k$ and obtain the expression

$$\alpha_1 \cdots \alpha_k = R_{p_1, A_1} \cdots R_{p_s, A_s}$$

for certain pairwise distinct p_1, \dots, p_s and pairwise disjoint A_1, \dots, A_s . Similarly, we can rearrange the factors in the product $\beta_1 \cdots \beta_l$ such that it equals $L_{q_1, B_1} \cdots L_{q_t, B_t}$ for certain pairwise distinct q_1, \dots, q_t and pairwise disjoint B_1, \dots, B_t . It follows that we can achieve the expression of the form (34) for α .

Suppose $p_i \in B_j$, for some $1 \leq i \leq k$ and $1 \leq j \leq l$. Rearranging, if necessary, the factors in α , we obtain an expression for α containing the factor $R_{p_i, A_i} L_{q_j, B_j}$. Then rearrange the factors constituting L_{q_j, B_j} in such a way that the obtained factorization of α contains the factor $R_{p_i, A_i} \lambda_{q_j, p_i}$. In view of (29) $\rho_{p_i, a} \lambda_{q_j, p_i} = \lambda_{q_j, p_i} e_a$. Applying this equality several times we obtain

$$R_{p_i, A_i} \lambda_{q_j, p_i} = \lambda_{q_j, p_i} \prod_{a \in A_i} e_a.$$

Applying (21) to the obtained expression for α we move all e_a -s to the right of all L_{q_j, B_j} -s. The resulting expression for α will be of the form (34),

but without the factor R_{p_i, A_i} , moreover possibly without some L_{q_j, B_j} -s and with a new E (containing more e_i -s). What we have reached is that the number of p_i -s contained in some B_j -s in the renewed expression for α is decreased by one. Applying the described rewriting several times we obtain an expression for α such that the condition (i) is satisfied.

Applying analogous manipulations the expression for α can be rewritten such that the condition (ii) is also satisfied.

Now we can assume that α is written in the form (34) and the conditions (i) and (ii) are satisfied. Suppose there is $a \in M$ such that $a = q_i$ or $a \in B_i$ for certain i . Rewrite the expression for α such that it contains the factor $L_{q_i, B_i} e_a$ and apply (21) several times. We will obtain

$$L_{q_i, B_i} e_a = e_{q_i} \prod_{b \in B_i} e_b.$$

This and inductive arguments on the number of factors of the form L_{q_i, B_i} in (34) show that α can be rewritten such that in the given expression for α the set M is disjoint with $\cup_{i=1}^l (\{q_i\} \cup B_i)$.

Let us continue the rewriting of the expression for α . Suppose there is $a \in M$ such that $a = p_i$ or $a \in A_i$ for certain i . We can assume that e_a is the first factor of E . In view of the first relation of (21) e_a commutes with every L_{q_i, B_i} . Hence we rewrite the expression such that e_a is located between the group of the factors R_{p_i, A_i} -s and the group of the factors L_{q_i, B_i} -s of our expression (34). Moreover, we can assume that this expression contains the factor $R_{p_i, A_i} e_a$. Similarly to as it was done previously we rearrange this factor and obtain

$$e_a R_{p_i, A_i} = e_{p_i} \prod_{x \in A_i} e_x.$$

Thus the number of such $a \in M$ that $a = p_i$ or $a \in A_i$ for certain i has been decreased by one. The difficulty here is that the current expression for α may be not of the form (34). To reach the expression of the required form we have to move the product $e_{p_i} \prod_{x \in A_i} e_x$ to the position to the right of all the L_{q_i, B_i} -s. It is enough to show that such a movement is possible for every e_x with $x \in \{p_i\} \cup A_i$ and apply induction. If $x \notin \cup_{i=1}^l (\{q_i\} \cup B_i)$ then e_x commutes with each L_{q_i, B_i} , and the required movement is performed. Otherwise, applying (possibly several times) (21) we obtain

$$e_x L_{q_i, B_i} = e_{q_i} \prod_{b \in B_i} e_b.$$

Since the sets $\{q_j\} \cup B_j$, $1 \leq j \leq l$, are pairwise disjoint it follows that every $y \in \{q_i\} \cup B_i$ does not belong to any of the sets $\{q_j\} \cup B_j$, $1 \leq j \leq l$,

$j \neq i$. Therefore, e_y commutes with all L_{q_j, B_j} -s, $j \neq i$, by (21). This completes the proof. \square

Denote $T = (\cup_{i=1}^k \{p_i\}) \cup (\cup_{i=1}^l \{q_i\})$, $s = |T|$. Enumerate the elements of T in some way, suppose $T = \{t_1, \dots, t_s\}$.

Define the sets C_i and $R_{C_i}^{t_i}$, $1 \leq i \leq s$, in the following way. Let $1 \leq i \leq s$.

- If $t_i = p_j$ for some j we set $C_i = A_j \cup \{p_j\}$ and $R_{C_i}^{t_i} = R_{p_j, A_j}$.
- If $t_i \notin \cup_{j=1}^k \{p_j\}$ we set $C_i = \{t_i\}$ and $R_{C_i}^{t_i} = e$.

The above defined sets C_1, \dots, C_s are pairwise disjoint, their union coincides with $T \cup (\cup_{i=1}^k A_i)$. Moreover, $t_i \in C_i$ for every possible i .

Similarly, define the sets $D_i, L_{D_i}^{t_i}$, $1 \leq i \leq s$:

- If $t_i = q_j$ for some j we set $D_i = B_j \cup \{q_j\}$ and $L_{D_i}^{t_i} = L_{q_j, B_j}$.
- If $t_i \notin \cup_{j=1}^l \{q_j\}$ we set $D_i = \{t_i\}$ and $L_{D_i}^{t_i} = e$.

The above defined sets D_1, \dots, D_s are pairwise disjoint, their union coincides with $T \cup (\cup_{i=1}^l B_i)$. Moreover, $t_i \in D_i$ for every possible i .

Corollary 2. *Every element of S can be presented in the form*

$$R_{C_1}^{t_1} \cdots R_{C_s}^{t_s} L_{D_1}^{t_1} \cdots L_{D_s}^{t_s} E_M \sigma, \quad (36)$$

where C_1, \dots, C_s are pairwise disjoint, D_1, \dots, D_s are pairwise disjoint, $t_i \in C_i \cap D_i$, $1 \leq i \leq s$, M is disjoint with $(\cup_{i=1}^s (C_i \cup D_i))$ and $\sigma \in \mathcal{S}_n$.

Proof. Follows from Proposition 3. \square

Let B be a subset of $\{1, \dots, n\}$. Denote by \mathcal{S}_B the subgroup of \mathcal{S}_n generated by all $\sigma_{i,j}$ with $i, j \in B$. Obviously, \mathcal{S}_B is isomorphic to $\mathcal{S}_{|B|}$.

Call an expression of the form (36) such that the conditions of Corollary 2 are satisfied a *canonical word*.

Let $F = M \cup (\cup_{i=1}^s (C_i \setminus D_i))$ and $G = \mathcal{S}_{D_1} \oplus \cdots \oplus \mathcal{S}_{D_s} \oplus \mathcal{S}_F$. The group G depends on $\{D_1, \dots, D_s\}, F$, but we do not indicate this into the notation just not to overload it.

Call two canonical words

$$R_{C_1}^{t_1} \cdots R_{C_s}^{t_s} L_{D_1}^{t_1} \cdots L_{D_s}^{t_s} E_{M_1} \sigma_1 \text{ and}$$

$$R_{C'_1}^{t'_1} \cdots R_{C'_s}^{t'_s} L_{D'_1}^{t'_1} \cdots L_{D'_s}^{t'_s} E_{M_2} \sigma_2$$

equivalent provided that there is a permutation $\tau \in \mathcal{S}_s$ such that $C_i = C'_{\tau(i)}$, $D_i = D'_{\tau(i)}$, $1 \leq i \leq s$, $M_1 = M_2$ and $\sigma_1 \sigma_2^{-1} \in G$.

Proposition 4. *If two canonical words are equivalent then their values in S are equal.*

The proof will be derived from a series of the following lemmas.

Lemma 7. *For pairwise distinct i, j, q*

$$\lambda_{q,i}\lambda_{q,j}\sigma_{i,j} = \lambda_{q,i}\lambda_{q,j} \quad (37)$$

Furthermore,

$$\lambda_{q,i}\lambda_{q,j}\sigma_{q,j} = \lambda_{q,i}\lambda_{q,j}. \quad (38)$$

Proof. Applying (25) and (9) we compute

$$\lambda_{q,i}\lambda_{q,j}\sigma_{i,j} = \lambda_{q,i}\lambda_{i,j}\sigma_{i,j} = \lambda_{q,i}\pi^{-1}\lambda_1\sigma_1\pi = \lambda_{q,i}\pi^{-1}\lambda_1\pi = \lambda_{q,i}\lambda_{i,j} = \lambda_{q,i}\lambda_{q,j},$$

where π is an element of \mathcal{S}_n such that $\pi(1) = i$ and $\pi(2) = j$. The relation (38) follows by the same argument. \square

Lemma 8. *Let $i \neq j$. Then*

$$e_i e_j \sigma_{i,j} = e_i e_j. \quad (39)$$

Proof. Applying consequently (28), $\rho_{i,j}\sigma_{i,j} = \rho_{j,i}$ (which holds by Lemma 2 and (9)), (28) and (21) we compute

$$e_i e_j \sigma_{i,j} = e_i \lambda_{i,j} \rho_{i,j} \sigma_{i,j} = \lambda_{j,i} \rho_{j,i} \lambda_{i,j} \rho_{j,i} = \lambda_{j,i} e_i e_j \rho_{j,i} = e_i e_j. \quad \square$$

Lemma 9. *Let $\alpha = R_{C_1}^{t_1} \cdots R_{C_s}^{t_s} L_{D_1}^{t_1} \cdots L_{D_s}^{t_s} E_M$. Then α is stabilized by G from the right, that is $\alpha\sigma = \alpha$ for every $\sigma \in G$.*

Proof. Suppose first that $\alpha \in \mathcal{S}_{D_r}$, $1 \leq r \leq s$. It is enough to consider only the case when $\alpha = \sigma_{i,j}$, $i, j \in D_r$. Since E_M commutes with $L_{D_r}^{t_r}$ we are only to establish that $L_{D_r}^{t_r} \sigma_{i,j} = L_{D_r}^{t_r}$. But this equality follows from (37) and (38).

Suppose now that $\alpha \in \mathcal{S}_F$. As in the previous paragraph, we consider only the case when $\alpha = \sigma_{i,j}$, $i, j \in F$. To prove the statement it is enough to show that

$$\alpha\sigma_{i,j} = \alpha e_i e_j \sigma_{i,j} \quad (40)$$

and apply (39).

If $i \in M$ then $\alpha e_i = \alpha$ by the definition of E_M and (14).

Suppose $i \in C_r \setminus D_r$ for some r , $1 \leq r \leq s$, and show that $\alpha e_i = \alpha$. Firstly, $E_M e_i = e_i E_M$ by (14). Further, $L_{D_j}^{t_j} e_i = e_i L_{D_j}^{t_j}$, $1 \leq j \leq s$, by the first relation of (21). Finally, $R_{C_r}^{t_r} e_i = R_{C_r}^{t_r}$ by the second relation of (22). This completes the proof. \square

Proof of Proposition 4. Suppose $R_{C_1}^{t_1} \cdots R_{C_s}^{t_s} L_{D_1}^{t_1} \cdots L_{D_s}^{t_s} E_M$ is a canonical word and $x_i \in C_i \cap D_i$, $1 \leq i \leq s$. In view of Lemma 1 and Corollary 1 it is enough to show that if we replace t_i -s by x_i -s, the obtained canonical word has the same value in S . Fix some index i . We can assume that the initial canonical word has the factor $\rho_{t_i, x_i} \lambda_{t_i, x_i}$. In view of (32) this factor equals $\rho_{x_i, t_i} \lambda_{x_i, t_i}$. It follows that $R_{C_i}^{t_i} L_{D_i}^{t_i} = R_{C_i}^{x_i} L_{D_i}^{x_i}$, which completes the proof. \square

5 Canonical form for the elements of \mathcal{PIP}_n

We start this section from introducing the notation for certain elements of the monoid \mathcal{PIP}_n . For distinct x and y of X we set

$$\begin{aligned} s_{x,y} &= \{ \{x, y'\}, \{x', y\}, \{t, t'\}_{t \in X \setminus \{x,y\}} \}, \\ r_{x,y} &= \{ \{x, y, x'\}, \{y'\}, \{t, t'\}_{t \in X \setminus \{x,y\}} \}, \\ l_{x,y} &= \{ \{x, x', y'\}, \{y\}, \{t, t'\}_{t \in X \setminus \{x,y\}} \} \text{ and} \\ \varepsilon_x &= \{ \{x\}, \{x'\}, \{t, t'\}_{t \in X \setminus \{x\}} \}. \end{aligned}$$

Furthermore, we set $s_i = s_{i,i+1}$, $r_i = r_{i,i+1}$ and $l_i = l_{i,i+1}$ for $1 \leq i \leq n-1$. The elements s_1, \dots, s_{n-1} generate the group of units of \mathcal{PIP}_n which is isomorphic to the symmetric group \mathcal{S}_n and will be identified with it.

We will use the following statement.

Proposition 5 ([KMai]). *Let $n \geq 3$. Then \mathcal{PIP}_n is generated by \mathcal{S}_n , r_1 and l_1 .*

Proposition 6. *The map from S to \mathcal{PIP}_n , sending σ_i to s_i , λ_i to l_i and ρ_i to r_i , $1 \leq i \leq n-1$, extends to an epimorphism $\varphi : S \rightarrow \mathcal{PIP}_n$.*

Proof. Firstly we make sure that $\varepsilon_i = l_{i-1} r_{i-1}$, $2 \leq i \leq n$, and $\varepsilon_1 = s_1 \varepsilon_2 s_1$. Then we verify that for the elements s_i , l_i , r_i and ε_i all the relations corresponding to the relations (1)-(17) hold. This and Proposition 5 imply the needed statement. \square

Some examples of the relations satisfied by the generating elements of the monoid \mathcal{PIP}_n , are given on Figures 3, 4 and 5.

Corollary 2 and Proposition 6 imply that every element of \mathcal{PIP}_n can be written as φ -image of some canonical word from S . Now we are going to show that such a presentation is unique.

Theorem 1. *The map $\varphi : S \rightarrow \mathcal{PIP}_n$ from Proposition 6 is an isomorphism.*

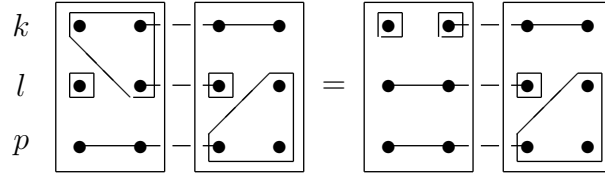


Figure 3: An illustration of the equality $l_{k,l}l_{p,l} = \epsilon_k l_{p,l}$.

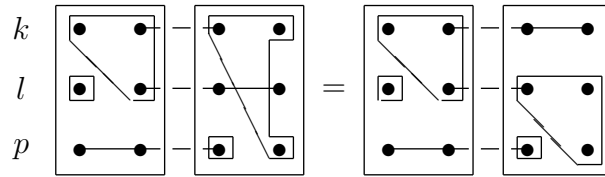


Figure 4: An illustration of the equality $l_{k,l}l_{k,p} = l_{k,l}l_{l,p}$.

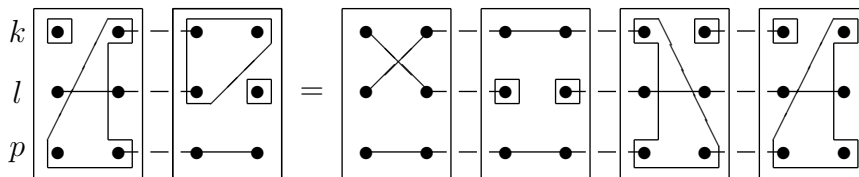


Figure 5: An illustration of the equality $l_{p,k}r_{k,l} = s_{k,l}\epsilon_l r_{p,k}l_{p,k}$.

Proof. We are to prove that the map φ is injective. Applying Proposition 4 it is enough to show that if φ -images of values of two canonical words in \mathcal{PIP}_n are equal then these canonical words are equivalent. For this, we compute the value of the image of a canonical word in \mathcal{PIP}_n . For the word (36) this is the element

$$\{(C_i \cup \sigma(D'_i))_{1 \leq i \leq s}, \{x\}_{x \in K_1}, \{\sigma(x')\}_{x \in K_2}, \{x, \sigma(x')\}_{x \in K_3}\},$$

where $K_1 = M \cup (\cup_{i=1}^s (D_i \setminus C_i))$, $K_2 = F = M \cup (\cup_{i=1}^s (C_i \setminus D_i))$, $K_3 = X \setminus ((\cup_{i=1}^s (C_i \cup D_i)) \cup M)$. The statement now follows from the definition of equivalent canonical words. \square

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