

On three approaches to conjugacy in semigroups

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Abstract

We compare three approaches to the notion of conjugacy for semigroups, the first one via the transitive closure of the $uv \sim vu$ relation, the second one via an action of inverse semigroups on themselves by partial transformations, and the third one via characters of finite-dimensional representations.

1 Introduction

Let G be a group. Then it acts on itself via conjugation, $x \mapsto gxg^{-1}$, $x, g \in G$; the orbits of this action define an equivalence relation on G and are called the *conjugacy classes*. There are alternative equivalent ways to define conjugate elements. To describe the first one we note that $y = gxg^{-1}$ can be written as $y = (gx)g^{-1}$ and in this case $x = g^{-1}(gx)$. Conversely, if $x = uv$ and $y = vu$ for some $u, v \in G$, then $y = vxv^{-1}$ and thus x and y are conjugate. Hence the elements x and y are conjugate in G if and only if there exist $u, v \in G$ such that $x = uv$ and $y = vu$.

Suppose now that the group G is finite and let $\varphi : G \rightarrow \mathbf{GL}(V)$ be a finite-dimensional complex representation of G . Then the traces of the linear operators $\varphi(x)$ and $\varphi(g)\varphi(x)\varphi(g)^{-1}$ coincide for all $x, g \in G$. Conversely, as the characters of irreducible representations of G form a basis in the space of functions constant on conjugacy classes, it follows that $x, y \in G$ are conjugate if and only if for each finite-dimensional representation φ of G the traces of the linear operators $\varphi(x)$ and $\varphi(y)$ coincide.

The aim of the present note, which is inspired by [Hi2], is to extend these results to some classes of semigroups. The most traditional approach to the notion of conjugacy for semigroups is the one using the so-called G -conjugacy, defined as follows: the elements x, y of a semigroup S , in which G is the group of units, are said to be G -conjugate, which is denoted by $x \sim_G y$, provided that $x = gyg^{-1}$ for some $g \in G$. This notion was studied from different

points of view by many authors, see for example [Pu, Li, KM1]. However, this approach is not unique. Another approach, which will be discussed in detail in the next section, comes from the equivalence relation generated by the (non-transitive in general) relation on S , which relates the element uv to the element vu for all $u, v \in S$. This notion, which has roots in the study of free monoids, see [La, 11.5], was also studied in [GK, KM1, KM2, Ku] for various special classes of semigroups. In the present paper we show that for some classes of semigroups the latter notion of conjugacy admits, just like for groups, alternative descriptions via the action of a semigroup on itself, and/or via characters of finite-dimensional representations.

As usually, we denote *Green's relations* on a semigroup by \mathcal{J} , \mathcal{D} , \mathcal{R} , \mathcal{L} and \mathcal{H} . If $\varphi : S \rightarrow \text{End}_{\mathbb{C}}(V)$ is a representation of a semigroup S , then the *character* of φ is the function $\chi_{\varphi} : S \rightarrow \mathbb{C}$ for which $\chi_{\varphi}(s)$ is defined as the trace of $\varphi(s)$, $s \in S$. A semigroup S is called *group-bound* provided that for each $x \in S$ there exists $k \in \{1, 2, \dots\}$ such that x^k lies in a subgroup of S . If S is group-bound, then for $x \in S$ we denote by e_x the idempotent of the subgroup containing x^k as above. For a partial transformation a we denote by $\text{dom}(a)$ and $\text{im}(a)$ the *domain* or the *image* of a respectively.

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2 Three approaches to the notion of conjugacy for semigroup

2.1 The first approach: via the transitive closure of the relation $uv \sim vu$

Let A be a non-empty alphabet and A^* the corresponding free monoid. Two elements $x, y \in A^*$ are called *conjugate* if there are $u, v \in A^*$ such that $x = uv$ and $y = vu$, or, equivalently, if there is $u \in A^*$ such that $ux = yu$ (see e.g. [La, 11.5]). These two relations, however, do not coincide on non-free semigroups in general (for example if S contains a zero element, then the second relation obviously degenerates). Hence if S is a semigroup, we

will call $x, y \in S$ *primarily conjugate* and denote this fact by $x \sim_p y$ if there are $u, v \in S^1$ such that $x = uv$ and $y = vu$. The relation \sim_p is reflexive and symmetric, while not transitive in general, in contrast to what one has for groups and free semigroups. Let \sim be the transitive closure of \sim_p . We call two elements $x, y \in S$ *conjugate* if $x \sim y$. It is easily verified that for monoids $x \sim_G y$ always implies $x \sim y$. The converse inclusion is not true in the general case, see for example [GK]. The relation \sim was studied for many classes of semigroup, see for example [La, GK, KM1, KM2, Ku].

2.2 The second approach: via an action of a semigroup on itself

Let S be an inverse group-bound semigroup, and \geq denote the natural partial order on S^1 , that is $a \geq b$ if and only if there is an idempotent $e \in S$ such that $b = ae$ (see also [Ho, 5.2] for alternative definitions). If $e, f \in S$ are idempotents then $e \geq f$ is equivalent to the equality $ef = fe = f$. Let $a \in S^1, x \in S$. Set

$$a \cdot x = \begin{cases} axa^{-1}, & \text{if } a^{-1}a \geq e_x; \\ \text{not defined,} & \text{otherwise.} \end{cases} \quad (1)$$

It is obvious that $x \mapsto a \cdot x, x \in S$, is a partial one-to-one transformation of S . Moreover, the following lemma shows that (1) in fact defines an action of S^1 on S .

Lemma 1. *Let $a, b \in S^1, x \in S$. Then $ba \cdot x$ is defined if and only if $a \cdot x$ and $b \cdot (a \cdot x)$ are defined. Moreover, in the case when $ba \cdot x$ is defined we have the equality $ba \cdot x = b \cdot (a \cdot x)$.*

Proof. Using the Preston-Wagner representation (see e.g. [CP, Theorem 1.20]) we can assume that the semigroup S is a subsemigroup of some inverse symmetric semigroup $\mathcal{IS}(X)$. For a group-bound element $a \in \mathcal{IS}(X)$ the element e_a is an idempotent acting identically on the set consisting of all $t \in X$ which belong to cycles of a (see [GM, 5.1] for the case of finite X , the case of infinite X can be treated easily using similar arguments). This set is called *the stable image* of a and is denoted by $\text{stim}(a)$ (see for example [GM]). The condition $a^{-1}a \geq e_x$ is then equivalent to $\text{dom}(a) \supseteq \text{stim}(x)$.

Observe that whenever the condition $\text{dom}(a) \supseteq \text{stim}(x)$ holds there is a bijection (given by $(t_1, \dots, t_l) \mapsto (a(t_1), \dots, a(t_l))$) from cycles of x to cycles of axa^{-1} . This, in particular, implies that $a(\text{stim}(x)) = \text{stim}(axa^{-1})$.

Suppose that $a, b, x \in \mathcal{IS}(X)$ are group-bound elements and that $a \cdot x$ and $b \cdot (a \cdot x)$ are defined. Since $a \cdot x$ is defined we have $\text{dom}(a) \supseteq \text{stim}(x)$,

which implies that

$$\text{im}(a) \supseteq a(\text{stim}(x)) = \text{stim}(axa^{-1}). \quad (2)$$

In addition, since $b \cdot (a \cdot x)$ is defined we have the inclusion

$$\text{dom}(b) \supseteq \text{stim}(axa^{-1}). \quad (3)$$

From (2) and (3) it follows that $\text{dom}(b) \cap \text{im}(a) \supseteq a(\text{stim}(x))$, whence

$$\text{dom}(ba) = a^{-1}(\text{dom}(b) \cap \text{im}(a)) \supseteq a^{-1}(a(\text{stim}(x))) = \text{stim}(x),$$

which means that $ba \cdot x$ is defined.

Suppose now that $ba \cdot x$ is defined. This means that $\text{dom}(ba) \supseteq \text{stim}(x)$ which implies that $a^{-1}(\text{im}(a) \cap \text{dom}(b)) \supseteq \text{stim}(x)$. This, in turn, gives us

$$\text{dom}(b) \cap \text{im}(a) \supseteq a(\text{stim}(x)). \quad (4)$$

Therefore, $\text{dom}(a) \supseteq \text{stim}(x)$, and thus $a(\text{stim}(x)) = \text{stim}(axa^{-1})$. This and (4) imply that $\text{dom}(b) \supseteq \text{stim}(axa^{-1})$, and hence both $a \cdot x$ and $b \cdot (a \cdot x)$ are defined.

That $ba \cdot x = b \cdot (a \cdot x)$ whenever $ba \cdot x$, $a \cdot x$ and $b \cdot (a \cdot x)$ are defined follows from the definition of our action and the fact that for an inverse semigroup the operation of taking the inverse of a given element is an anti-involution. \square

Let S be an inverse semigroup and $x, y \in S$. Set $x \approx_p y$ if there is some $a \in S^1$ such that $y = a \cdot x$ or $x = a \cdot y$. This relation is reflexive because $x = 1 \cdot x$. It is also symmetric by definition. Let \approx be the transitive closure of \approx_p . We will call this relation *the relation of conjugacy in the action sense*.

2.3 The third approach: via characters of finite-dimensional representations

As we have mentioned in the Introduction, two elements x, y of a finite group G are conjugate if and only if for every finite-dimensional complex representation φ of G the equality $\chi_\varphi(x) = \chi_\varphi(y)$ holds. Let now S be a semigroup and $x, y \in S$. We will call x and y *conjugate in the character sense* and denote this fact by $x \equiv y$ provided that for every finite-dimensional complex representation φ of S we have the equality $\chi_\varphi(x) = \chi_\varphi(y)$.

3 Comparing the three approaches

Theorem 2. *Let S be a regular group-bound semigroup with finite \mathcal{D} -classes. Then for $x, y \in S$ we have $x \sim y$ if and only if $x \equiv y$.*

Proof. If $a, b \in S$ are such that $a \sim_p b$, then $a = uv$ and $b = vu$ for some $u, v \in S$. Let φ be a finite dimensional complex representation of S . Then the traces of the linear operators $\varphi(u)\varphi(v)$ and $\varphi(v)\varphi(u)$ are equal (by a standard exercise in linear algebra), and thus the necessity of the claim follows.

Suppose now that $a, b \in S$ and $a \equiv b$. Since S is regular and group-bound, from [Ku, Corollary 6] for all $x, y \in S$ it follows that $x \sim y$ if and only if $xe_x \sim ye_y$. Having this in mind, it is enough to prove that $ae_a \sim be_b$.

Let us first show that $a \equiv ae_a$ and $b \equiv be_b$. It is of course enough to prove the first formula. Let $\varphi : S \rightarrow \text{End}_{\mathbb{C}}(V)$ be a finite-dimensional representation of S . For $t \in \mathbb{C}$ set $V_t = \{v \in V : (\varphi(a) - t)^{\dim(V)}v = 0\}$. From the definition it follows that the linear operator $\varphi(e_a)$ is the projection of V onto $U = \bigoplus_{t \neq 0} V_t$ with the kernel V_0 . In particular, the actions of $\varphi(a)\varphi(e_a)$ and $\varphi(a)$ on U coincide, and both $\varphi(a)\varphi(e_a)$ and $\varphi(a)$ act nilpotently on V_0 . This implies that $\chi_{\varphi}(a) = \chi_{\varphi}(ae_a)$ and hence $a \equiv ae_a$. As a consequence we also obtain $ae_a \equiv be_b$.

Let us now show that the elements ae_a and be_b belong to the same \mathcal{D} -class of S . To do this we have to recall the construction of some induced modules for semigroups, which follows closely [CP, 5.4].

For $x \in S$ denote by D_x , L_x , and H_x the \mathcal{D} -class, \mathcal{L} -class or \mathcal{H} -class of x , respectively. Let $e \in S$ be an idempotent and $\varphi : H_e \rightarrow \mathbf{GL}(W)$ be a finite-dimensional representation of the maximal subgroup H_e . Let $H_1 = H_e$, H_2, \dots, H_k be the list of all \mathcal{H} -classes in L_e . Fix some a_i in each H_i such that a_i is an idempotent if H_i is a subgroup.

Let $s \in S$ and $i \in \{1, 2, \dots, k\}$. Assume that $sa_i \in D_e$ and let a'_i be some inverse to a_i (which exists as S is regular). Then $sa_i = (sa_i a'_i)a_i$, where $sa_i a'_i$ must belong to D_e as well. As D_e is finite, from [Ho, Lemma 1.3.3] it follows that $sa_i \in L_e$. This implies that we have exactly two possibilities: either $sa_i \notin L_e$ (and hence $sa_i \notin D_e$) or $sa_i = a_j s'$ for some uniquely determined $j \in \{1, 2, \dots, k\}$ and $s' \in H_e$. For $i \in \{1, 2, \dots, k\}$ let $W^{(i)}$ denote a copy of W . Then we can consider the vector space $\overline{W} = \bigoplus_{i=1}^k W^{(i)}$ and for every $s \in S$ define a linear operator on \overline{W} as follows: for $v \in W^{(i)}$ set

$$sv = \begin{cases} 0, & sa_i \notin L_e; \\ s'v \in W^{(j)}, & sa_i = a_j s' \text{ as above.} \end{cases} \quad (5)$$

This defines a representation $\overline{\varphi} : S \rightarrow \text{End}_{\mathbb{C}}(\overline{W})$. Observe that, by [CP, Theorem 2.17], for $s \in H_e$ we have $sa_i \in L_e$ if and only if a_i is an idempotent.

Moreover, in this case for a_j and s' from (5) we have $a_j = a_1 = e$ and $s' = s$ by [CP, Lemma 2.14]. This implies that for $s \in H_e$ we have $\chi_{\overline{\varphi}}(s) = m\chi_{\varphi}(s)$, where $m > 0$ is the number of idempotents in L_e .

Suppose that ae_a and be_b do not belong to the same \mathcal{D} -class of S . Then they do not belong to the same \mathcal{J} -class either for on group bound semigroups Green's relations \mathcal{D} and \mathcal{J} coincide, see [Hi1, Theorem 1.2.20]. Hence, without loss of generality we can assume that $ae_a \notin Sbe_bS$. The element $ae_a \in H_{e_a}$ is a group element. Let $\varphi : H_{e_a} \rightarrow \mathbf{GL}(W)$ be some finite-dimensional complex representation such that $\chi_{\varphi}(ae_a) \neq 0$. Such representation exists since characters of irreducible representations form a basis in the space of class functions. Then $\chi_{\overline{\varphi}}(ae_a) = m\chi_{\varphi}(ae_a) \neq 0$, while $\overline{\varphi}(be_b) = 0$ as $ae_a \notin Sbe_bS$. This contradicts $ae_a \equiv be_b$ and proves that the elements ae_a and be_b do belong to the same \mathcal{D} -class of S .

Since $ae_a\mathcal{D}be_b$ and both are group elements, from [CP, Theorem 2.20] it follows that there exist a pair, t and t' , of mutually inverse elements in the same \mathcal{D} -class such that $tbe_b t' \mathcal{H} ae_a$. Let ψ be a finite-dimensional representation of S . Then we have

$$\chi_{\psi}(tbe_b t^{-1}) = \chi_{\psi}(be_b t^{-1}t) = \chi_{\psi}(be_b) = \chi_{\psi}(ae_a), \quad (6)$$

where the second equality follows from $be_b t^{-1}t = be_b$, which, in turn, follows from [CP, Lemma 2.14 and Theorem 2.17]. In particular, the characters of $tbe_b t^{-1}$ and ae_a coincide in all cases when $\psi = \overline{\varphi}$, where φ is an irreducible representation of H_{ae_a} . As on elements from H_{ae_a} the character of $\overline{\varphi}$ differs from that of φ only by a non-zero constant (the number of idempotents in L_{ae_a} , see above), it follows that the characters of $tbe_b t^{-1}$ and ae_a coincide for every irreducible representation of H_{ae_a} . Therefore $tbe_b t^{-1}$ and ae_a are conjugate as elements of H_{ae_a} , in particular $tbe_b t^{-1} \sim ae_a$. As $be_b = (be_b t^{-1})t$, the elements be_b and $tbe_b t^{-1}$ are primarily conjugate. This implies that $ae_a \sim be_b$ and completes the proof. \square

Theorem 3. *Let S be a group-bound inverse semigroup and $x, y \in S$. Then the following conditions are equivalent:*

- (a) $x \sim y$;
- (b) $x \approx y$;
- (c) there is $z \in S$ such that $z = a \cdot x = b \cdot y$ for some $a, b \in S^1$.

Proof. The implication (c) \Rightarrow (b) is obvious.

Let us prove the implication (a) \Rightarrow (c). Using the Preston-Wagner representation we again think of S as of a group-bound subsemigroup of the

inverse symmetric semigroup $\mathcal{IS}(X)$. Suppose that $x, y \in S$ are such that $x \sim y$. We take $z = xe_x$. First we observe that $\text{dom}(e_x) = \text{stim}(x)$, and thus $e_x \cdot x$ is defined. Now we observe that $e_x \cdot x = e_xxe_x = xe_x$. Therefore $x \approx xe_x$ and we can take $a = e_x$. Analogously one shows that $e_y \cdot y = ye_y$.

Since $x \sim y$ it follows from [Ku, Corollary 6] that $xe_x \sim ye_y$ and $xe_x\mathcal{D}ye_y$. By [CP, Theorem 2.20] there exists $t \in L_{e_y} \cap R_{e_x}$ such that $xe_x = tye_yt^{-1}$. Since $\text{dom}(t) = \text{dom}(e_y) = \text{stim}(ye_y)$, it follows that $t \cdot ye_y$ is defined and equals xe_x . Hence $te_y \cdot y$ is defined and equals xe_x by Lemma 1. We take $b = te_y$ and the implication (a) \Rightarrow (c) follows.

Finally, we show that (b) \Rightarrow (a). Again, using the Preston-Wagner representation we think of S as of a group-bound subsemigroup of the inverse symmetric semigroup $\mathcal{IS}(X)$. Let $x, y \in S$ be such that $y = a \cdot x = axa^{-1}$ for some $a \in S$ such that $a^{-1}a \geq e_x$. Then $\text{dom}(a) \supseteq \text{stim}(x)$ and it follows that a induces a bijection between $\text{stim}(x)$ and $\text{stim}(y)$. Let $\hat{a} = ae_x \in S$ denote the restriction of a to $\text{stim}(x)$. Then we have $\hat{a}^{-1}\hat{a} = e_x$, $\hat{a}\hat{a}^{-1} = e_y$ by definition. Moreover, it also follows that $ye_y = \hat{a}xe_x\hat{a}^{-1}$ and $xe_x = \hat{a}^{-1}ye_y\hat{a}$. Hence

$$xe_x = \hat{a}^{-1}ye_y\hat{a} \sim ye_y\hat{a}\hat{a}^{-1} = ye_y.$$

Applying [Ku, Corollary 6] we obtain $x \sim y$, which completes the proof. \square

From Theorems 2 and 3 we immediately obtain the following corollaries.

Corollary 4. *Let S be a finite regular semigroup and $a, b \in S$. Then $a \sim b$ if and only if $a \equiv b$.*

Corollary 5. *Let S be a finite inverse semigroup and $a, b \in S$. Then $a \sim b$ if and only if $a \approx b$ if and only if $a \equiv b$.*

It would be interesting to extend the definition of the relation \approx to other classes of semigroups. We finish the paper with a an example, which shows that Theorem 2 is not true for non-regular semigroups.

Example 6. Let $S = \langle x : x^2 = x^3 \rangle$. Then the relation \sim on S is trivial (i.e. all conjugacy classes contain exactly one element). At the same time if $\varphi : S \rightarrow \text{End}_{\mathbb{C}}(V)$ is a finite-dimensional complex representation of S , then $\varphi(x^2)$ is a projection (as x^2 is an idempotent), so the same arguments as in the proof of Theorem 2 show that $\chi_{\varphi}(x) = \chi_{\varphi}(x^2)$, implying $x \equiv x^2$.

References

- [CP] A.H. Clifford, G.B. Preston, The algebraic theory of semigroups, vol I. Amer. Math. Soc. Surveys **7** (Providence, R.I.) 1961.

- [GK] O. Ganyushkin, T. Kormysheva, The chain decomposition of partial permutations and classes of conjugate elements of the semigroup \mathcal{IS}_n . Visnyk of Kyiv University, 1993, no.2, 10–18.
- [GM] O.Ganyushkin, V. Mazorchuk, Introduction to classical finite transformation semigroups. Preprint, available online at:
<http://www.math.uu.se/~mazor/PREPRINTS/SEMI/book.pdf>
- [Hi1] P. Higgins, Techniques of semigroup theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.
- [Hi2] P. Higgins, The semigroup of conjugates of a word. Internat. J. Algebra Comput. **16** (2006), no. 6, 1015–1029.
- [Ho] J. Howie, Fundamentals of semigroup theory. London Mathematical Society Monographs, New Series **12**. Oxford University Press, 1995.
- [Ku] G. Kudryavtseva, On conjugacy in regular epigroups, Preprint arXiv: math/0605698
- [KM1] G. Kudryavtseva, V. Mazorchuk, On conjugation in some transformation and Brauer-type semigroups. Publ. Math. Debrecen **70** (2007), no.1–2, 19–43.
- [KM2] G. Kudryavtseva, V. Mazorchuk, On the semigroup of square matrices, Preprint arXiv: math/0510624, to appear in Algebra Colloq.
- [La] G. Lallement, Semigroups and combinatorial applications. Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, New York-Chichester-Brisbane, 1979.
- [Li] S. Lipscomb, Symmetric inverse semigroups. Mathematical Surveys and Monographs **46**. American Mathematical Society, Providence, RI, 1996.
- [Pu] M. Putcha, Hecke algebras and monoid conjugacy classes. J. Algebra **173** (1995), no. 3, 499–517.

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